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Explosive resonant interaction of baroclinic Rossby waves and stability of multilayer quasi-geostrophic flow

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The amplitude equations governing the nonlinear interaction among normal modes are derived for a multilayer quasi-geostrophic channel. The set of normal modes can represent any wavy disturbance to a parallel shear flow, which may be stable or unstable. Orthogonality in the sense of pseudomomentum or pseudoenergy is used to obtain the amplitude equations in a direct fashion, and pseudoenergy and pseudomomentum conservation laws permit the properties of the interaction coefficients to be deduced. Particular attention is paid to triads exhibiting explosive resonant interaction, as they lead to nonlinear instability of the basic flow. The relationship between this mechanism and the most recently discovered nonlinear stability conditions is discussed.

Situations in which the basic velocity is constant in each layer are treated in detail. A particular formulation of the stability condition is given that emphasizes the close connection between linear and nonlinear stability. It is established that this stability condition is also a necessary condition: when it is not satisfied, and when the flow is linearly stable, explosive resonant interaction of baroclinic Rossby waves acts as a destabilizing mechanism. Two- and three-layer models are specifically considered; their stability features are presented in the form of stability diagrams, and interaction coefficients are calculated in particular cases.

1. Introduction

Baroclinic instability has long been recognized as playing a dominant role in both atmospheric and oceanic circulation, and has been one of the most widely studied phenomena in geophysical fluid dynamics. A variety of models have been analysed, initially by linear theory and then by weakly nonlinear theory. The nonlinear studies often address the evolution of an unstable mode in a slightly supercritical flow and predict its final stabilization. Quasi-geostrophic models, and layered models in particular, have been much used in this respect, owing to their great simplicity (see Pedlosky 1987 and references therein). Recently, such models have received renewed attention thanks to properties derived from energy-Casimir and momentum-Casimir conservation laws. Two related results have essentially been obtained: sufficient (Arnol'd-like) conditions of nonlinear stability (Shepherd 1988, 1993; Ripa 1991, 1992, 1993; Mu Mu 1991; Mu Mu *et al.* 1994; see also McIntyre & Shepherd 1987), and explicit bounds on the disturbance energy and potential enstrophy (Shepherd 1988, 1993; Mu Mu 1991; Mu Mu *et al.* 1994). Although those bounds exist only when stability is proved, they have been used by Shepherd (1988, 1993) to investigate the saturation of unstable flows.

In general, comparison between the nonlinear stability conditions and the linear (normal modes) instability conditions reveals that they do not coincide everywhere in parameter space. Two explanations can be advanced: (*a*) the nonlinear stability conditions, being only sufficient, could still be improved (as has been the case for the papers of Mu Mu 1991, Ripa 1992, 1993; see Mu Mu *et al.* 1994); (*b*) there exist nonlinear mechanisms capable of amplifying infinitesimal initial disturbances. The latter explanation should not be surprising: it is well known that the stability of a linearized system does not imply the stability of the original nonlinear system, even to arbitrarily small disturbances. Furthermore, such a mechanism has been pointed out in layered models of irrotational flows: Cairns (1979) and Craik & Adam (1979) have shown that a resonant triad of gravity waves can extract energy from the basic flow so that the three triad members grow simultaneously. This process, called explosive resonant interaction (ERI), is intimately related to the concept of negative wave (pseudo-) energy (see Cairns 1979; Craik 1985; Ripa 1990) and has been proved to render Cairns's three-layer model unstable in linearly subcritical conditions.

In the context of quasi-geostrophic flows, ERI has seldom been mentioned, although a number of papers deal with nonlinear interaction between Rossby waves (see Jones 1979*a*, *b*). Nevertheless, Romanova (1987) has addressed the question using a three-layer model, with constant velocity in each layer, and has shown that IRE may occur provided that the system is linearly unstable. Her model is infinite in the meridional direction and the linear stability condition coincides with the nonlinear condition obtained through the analogue of Arnol'd's first theorem. However, boundary constraints are known to stabilize quasi-geostrophic flows by imposing a minimum scale on the disturbance (see McIntyre & Shepherd 1987, §6). A finite channel width thus introduces new stability conditions whose nonlinear version is directly related to Arnol'd's second theorem (Mu Mu 1991; Ripa 1992, 1993; Mu Mu *et al.* 1994). The present paper specifically discusses that point in connection with the ERI of baroclinic Rossby waves. In particular, we show that Romanova's three-layer model can be made linearly stable by imposing boundary constraints, while it remains nonlinearly unstable. In that case, ERI provides the necessary mechanism to make disturbances grow.

The possibility of ERI within a triad depends on the sign of the interaction coefficients which parameterize the coupled influence of two waves on the third one. These coefficients can be directly related to the quadratic invariants corresponding to each wave, this relation leading to a simple criterion for ERI (see Craik 1985). In most previous works, the conserved quantity is the wave energy, but it seems more general to discuss the problem in terms of pseudoenergy and pseudomomentum, following Becker & Grimshaw (1993) and Vanneste & Vial (1995, hereafter referred to as VV). (Ripa 1990 gives an illuminating comparison between the concepts of wave energy (momentum) and pseudoenergy (pseudomomentum) in various frameworks.) To obtain the interaction coefficients, one usually introduces a multiple scale expansion, either in the basic equations or in a (Lagrangian) variational principle. This technique requires a large amount of algebra and does not seem very suitable for the general multilayer model we consider. Here, we adapt for that model the method described in VV, i.e. the rigorous expansion of the basic equations in normal modes. Orthogonality in the sense of pseudomomentum or pseudoenergy (Held 1985) is used to derive the evolution equations for the mode amplitudes. The properties of the interaction coefficients are directly deduced from the conservation laws of pseudomomentum and pseudoenergy, without requiring manipulation of the explicit expression of the coefficients. We introduce an important improvement to VV: the expansion remains valid for linearly unstable basic flows, the modes involved being stable, unstable or marginally stable. It is essential to emphasize that the method is useful for a wide class of models, as illustrated by the different physical systems considered in VV and in this paper. Our approach is also closely related to that of Romanova (1992), who uses an explicit Hamiltonian formulation to work out the normal modes expansion. However, her study concerns mainly the linear part of the equations.

In this paper, we develop a general formulation of the problem of nonlinear interaction of baroclinic Rossby waves propagating in a parallel shear flow. We use the quasi-geostrophic N-layer model, whose governing equations are briefly outlined in §2. The normal modes of the linearized system are presented in §3. We also discuss the orthogonality relations in the sense of pseudomomentum and pseudoenergy and deduce a consequence of the vanishing of the pseudomomentum for unstable modes. In §4, we expand the nonlinear equations in a series of normal modes, and obtain the evolution equations for their amplitudes. Particular attention is paid to marginally stable modes whose amplitude equations remain coupled. In §5, conservation of pseudomomentum and pseudoenergy is used to derive relations between the interaction coefficients. Such relations are particularly useful since they permit the linking of the three coefficients appearing in the resonant triad equations to a single constant. With the aid of these results, we discuss a generalization of Hasselmann's criterion for wave instability through nonlinear interaction. The condition of existence of ERI is then stated in terms of the frequency and pseudoenergy of the modes and, equivalently, in terms of their wavenumber and pseudomomentum. In §6, we restrict our attention to what we shall call generalized Phillips models, namely multilayer models with constant velocity in each layer. We first give a particular form of the dispersion relation analogous to Romanova's (1992) that is related to the pseudomomentum. The close connection between the sufficient stability condition due to Mu Mu *et al.* (1994) and the linear eigenvalue problem is pointed out. We derive a simplified version of Mu Mu *et al.*'s criterion that allows us to show that linear and nonlinear stability are equivalent in wide regions of the parameter space. However, when a linearly stable system cannot be proved nonlinearly stable, it appears to be unstable through ERI. The two- and three-layer models are then studied separately and some numerical results for the interaction coefficients are presented. Finally, §7 is devoted to a concluding discussion.

2. Basic equations and stability of the N-layer model

We consider the multilayer quasi-geostrophic model described for instance in Pedlosky (1987, §6.16) and used by Mu Mu (1991), Ripa (1992) and Mu Mu *et al.* (1994) for their stability analysis. The notation of the latter paper is largely followed here. The buoyany jumps $g' = g(\rho_{i+1}-\rho_i)/\rho_0$ are taken equal in each layer i so that the rotational Froude numbers

$$F_i = f_0^2 L^2/(g' d_i)$$

satisfy $F_i d_i = \text{const}$, $i = 1, N$. Here f_0 is the mean Coriolis parameter, ρ_0 the reference density, L a characteristic horizontal scale and d_i the depth of the layer i. Basically, the governing equations express the conservation in each layer of the potential vorticity given by

$$P_i = \nabla^2 \Phi_i + F_i \sum_{j=1}^{N} T_{ij} \Phi_j + \beta y,$$

where Φ_i is the streamfunction, $\beta = \beta_0 L^2/U$ is curvature parameter, scaled using a reference velocity U, and where we have introduced the tridiagonal matrix

$$\boldsymbol{T} = \begin{pmatrix} -1 & 1 & & & \\ 1 & -2 & 1 & & \\ & \cdot & \cdot & \cdot & \\ & & \cdot & \cdot & \cdot \\ & & & 1 & -1 \end{pmatrix},$$

corresponding to rigid vertical boundaries. A particular solution of the evolution equations is given by the steady parallel flow $U_i(y) = -\mathrm{d}\Psi_i(y)/\mathrm{d}y$, balanced by the slope of the interfaces. We are interested in the evolution of small-amplitude disturbances to this basic flow and we introduce the classical decomposition

$$\Phi_i = \Psi_i + \psi'_i \quad \text{and} \quad P_i = Q_i + q'_i,$$

and

$$q'_i = \nabla^2 \psi'_i + F_i \sum_{j=1}^{N} T_{ij} \psi'_j.$$

The dimensionless evolution equations for the disturbance are then given by

$$(\partial_t + U_i \partial_x) q'_i + Q_{iy} \partial_x \psi'_i + J(\psi'_i, q'_i) = 0, \tag{2.1}$$

with $Q_{iy} = \mathrm{d}Q_i/\mathrm{d}y$.

The invariants of (2.1), connected to its symmetry in t and x, constrain the dynamics of the system. Among them, pseudoenergy and pseudomomentum, which are quadratic to lowest order, are the most useful. Their finite-amplitude expressions have been deduced by Ripa (1992) and Mu Mu *et al.* (1994) (see also Shepherd 1988, 1993) and their density can be written

$$E = \sum_{i=1}^{N} d_i \left\{ \tfrac{1}{2} |\psi'_i|^2 + \int_0^{q'_i} [\Psi_i(Q_i + \tilde{q}) - \Psi_i(Q_i)] \,\mathrm{d}\tilde{q} \right\} + \sum_{i=1}^{N-1} \tfrac{1}{2} d_i F_i (\psi'_{i+1} - \psi'_i)^2, \tag{2.2a}$$

$$P = -\sum_{i=1}^{N} d_i \int_0^{q'_i} [Y_i(Q_i + \tilde{q}) - Y_i(Q_i)] \,\mathrm{d}\tilde{q}, \tag{2.2b}$$

where $\Psi_i(Q_i)$ and $Y_i(Q_i)$ are the inverse functions of Q_i (Ψ_i) and $Q_i(y)$, respectively.

It is now convenient to introduce a weighted streamfunction and potential vorticity, namely

$$\psi_i = F_i^{-1/2} \psi'_i \quad \text{and} \quad q_i = F_i^{-1/2} q'_i.$$

Their governing equations can then be cast in a vector form

$$(\partial_t + \boldsymbol{U} \partial_x) \boldsymbol{q} - \boldsymbol{L} \partial_x \boldsymbol{\psi} + \boldsymbol{S}(\boldsymbol{\psi}, \boldsymbol{q}) = 0, \tag{2.3}$$

where $\boldsymbol{\psi} = (\psi_i)^T$ and $\boldsymbol{q} = (q_i)^T$. These two vectors are related by

$$\boldsymbol{q} = \boldsymbol{D}\boldsymbol{\psi},$$

with $\boldsymbol{D} \equiv \boldsymbol{I}\nabla^2 + \boldsymbol{KTK}$, where $\boldsymbol{I}$ is the identity matrix and $\boldsymbol{K} \equiv \mathrm{diag}(F_i^{1/2})$. Owing to the particular definition of the streamfunction and potential vorticity vectors $\boldsymbol{\psi}$ and $\boldsymbol{q}$, $\boldsymbol{D}$ is a Hermitian operator with respect to the inner product $\langle \boldsymbol{q}, \boldsymbol{p} \rangle = \int\int_S \boldsymbol{q}^* \boldsymbol{p} \,\mathrm{d}s$, a property which will prove useful throughout this paper. The matrices $\boldsymbol{U}$ and $\boldsymbol{L}$ depend only on the basic flow and are given by

$$\boldsymbol{U} \equiv \mathrm{diag}(U_i) \quad \text{and} \quad \boldsymbol{L} \equiv \mathrm{diag}(-Q_{iy}),$$

while $\boldsymbol{S}(\boldsymbol{\psi}, \boldsymbol{q})$ is the vector of the nonlinear terms, with components

$$S_i = F_i^{1/2} J(\psi_i, q_i).$$

We consider the flow in a channel which is infinite in the x-direction and is of dimensional width lL in the y-direction, so that the boundary conditions read

$$\psi_x = 0 \quad \text{and} \quad \partial_t \int \psi_y \, \mathrm{d}x = 0 \quad \text{for} \quad y = 0, l. \tag{2.4}$$

The quadratic parts of the pseudoenergy and of the pseudomomentum are the conserved densities of the linearized equations. They will be used in the following. They can be deduced from (2.2) or by direct manipulation of the linearized version of (2.3), and take a simple form

$$E^{(2)} = -\frac{1}{2}\int_S (\boldsymbol{\psi}^T \boldsymbol{q} - \boldsymbol{q}^T \boldsymbol{U}\boldsymbol{L}^{-1}\boldsymbol{q})\,\mathrm{d}s, \quad P^{(2)} = \frac{1}{2}\int_S \boldsymbol{q}^T \boldsymbol{L}^{-1}\boldsymbol{q}\,\mathrm{d}s, \tag{2.5a, b}$$

with

$$\int_S (\cdot)\,\mathrm{d}s = \lim_{X\to\infty} \frac{1}{2X}\int_{-X}^{X} \mathrm{d}x \int_0^l \mathrm{d}y(\cdot).$$

The multilayer quasi-geostrophic model has recently received much attention since it has been found that analogues of Arnol'd's stability theorems can be deduced to prove its nonlinear (normed) stability. The first theorem (Holm *et al.* 1985; Ripa 1992) is based on the positive definiteness of $E - \alpha P$, and guarantees stability if there exists any value of α such that

$$Q_{iy}^{-1}(U_i - \alpha) < 0, \quad i = 1, N. \tag{2.6}$$

The analogue of Arnol'd's second theorem has led to several sufficient conditions, among which the strongest is due to Mu Mu *et al.* (1994, see also Ripa 1992). Nonlinear stability is guaranteed if one can find constant γ_i and α such that

$$0 < \gamma_i < -\frac{\mathrm{d}}{\mathrm{d}Q_i}(\Psi_i + \alpha y) < \infty, \quad i = 1, N \tag{2.7a}$$

and that

$$\boldsymbol{C} - (K_0 \boldsymbol{I} - \boldsymbol{K}\boldsymbol{T}\boldsymbol{K})^{-1} \tag{2.7b}$$

is a definite positive matrix. In the above, $\boldsymbol{C} \equiv \mathrm{diag}\,(\gamma_i)$, and K_0 is the eigenvalue of a differential problem ($K_0 = (\pi/l)^2$ is the channel model with rigid vertical boundaries).

3. Normal modes and orthogonality relations

We now turn to the analysis of the normal modes of the system which are solutions of the linearized version of (2.3). The linearized version of (2.3) can be written in the form

$$\partial_t \boldsymbol{L}^{-1}\boldsymbol{q} + \boldsymbol{N}\partial_x \boldsymbol{q} = 0, \tag{3.1}$$

where $\boldsymbol{N} \equiv \boldsymbol{L}^{-1}\boldsymbol{U} - \boldsymbol{D}^{-1}$ involves the non-local operator $\boldsymbol{D}^{-1}$ which is defined with homogeneous boundary conditions. Since $\boldsymbol{D}$ is Hermitian, and $\boldsymbol{U}$ and $\boldsymbol{L}$ are diagonal real matrices, $\boldsymbol{N}$ is also a Hermitian operator. For a particular zonal wavenumber k_a, the solution of (3.1) can be found in the form $\boldsymbol{q} = \boldsymbol{q}_a(y)\exp[\mathrm{i}k_a(x - c_a t)]$ by solving the eigenvalue problem

$$\boldsymbol{N}_a \boldsymbol{q}_a(y) = c_a \boldsymbol{L}^{-1}\boldsymbol{q}_a(y), \tag{3.2}$$

with

$$\boldsymbol{q}_a(0) = \boldsymbol{q}_a(l) = 0.$$

We denote by $\boldsymbol{N}_a$ the operator $\boldsymbol{N}$ where the substitution $\partial_x \leftarrow \mathrm{i}k_a$ has been made. The subscript a of the eigenvector $\boldsymbol{q}_a$ and of the eigenvalue c_a denotes the double dependence on (k_a, n_a), where n_a is used to distinguish the different solutions to (3.2) corresponding to the same k_a. The spectrum of the eigenvalues is discrete only if the basic velocity is constant in each layer; otherwise it has continuous parts, with c_a in any of the ranges $[\min_y U_i, \max_y U_i]$, $i = 1, N$. In these continuous parts, the eigenvector $\boldsymbol{q}_a$ possesses a singularity at values of y such that $U_i(y) = c_a$. Note that the boundary conditions prescribe vanishing circulation for the disturbance.

As $\boldsymbol{N}_a$ is Hermitian, orthogonality relations between the normal modes are easily derived. First, two solutions of (3.1) with different zonal wavenumbers are always orthogonal in the sense that the mean of their product (along the infinite length of the channel) vanishes. We can thus concentrate on the y dependence and consider two eigenvectors $\boldsymbol{q}_a$ and $\boldsymbol{q}_b$ with the same wavenumber $k_a = k_b$ so that $\boldsymbol{N}_a = \boldsymbol{N}_b$. The first orthogonality relation is deduced from (3.2) by noting that

$$[(c_a)^* - c_b]\{\boldsymbol{q}_a, \boldsymbol{L}^{-1}\boldsymbol{q}_b\} = 0,$$

where the inner product of two vectors is defined by

$$\{\boldsymbol{q}, \boldsymbol{p}\} = \int_0^l \boldsymbol{q}^* \boldsymbol{p}\, \mathrm{d}y.$$

Since $\boldsymbol{N}_a$ and $\boldsymbol{L}$ are real operators, the orthogonality relation can be explicitly written as

$$\{\boldsymbol{q}_a, \boldsymbol{L}^{-1}\boldsymbol{q}_b\} = \delta_{a^*, b}\, \tilde{P}_b, \tag{3.3}$$

where $\delta_{a^*,b} = 1$ if $(\boldsymbol{q}_a)^* = \boldsymbol{q}_b$, and $\delta_{a^*,b} = 0$ otherwise. As can be seen by comparing (2.5*b*) and (3.3), the latter expression extends to the multilayer model the orthogonality relation (in the sense of pseudomomentum) first introduced by Held (1985; see also VV). In (3.3), we use the notation $\tilde{P}_b$ to emphasize that the interpretation in terms of pseudomomentum differs according to the stability of mode b. A stable mode s for which c_s, and therefore $\boldsymbol{q}_s$ are real, corresponds to a Rossby wave modified by the basic shear. It contributes to the pseudomomentum in an isolated way and we can refer to this contribution as the wave pseudomomentum $P_s \equiv \tilde{P}_s$. On the other hand, an unstable mode u, for which c_u has a non-zero imaginary part, contributes to the pseudomomentum only through its coupling with the mode u^* corresponding to the conjugate phase velocity $(c_u)^*$. We can denote $R_u \equiv \tilde{P}_u$ the contribution of such a growing–decaying pair of modes. That an isolated growing mode has vanishing pseudomomentum is directly related to the fact that the pseudomomentum is a conserved quadratic quantity (Held 1985): in some sense, a mode can be unstable and grow only if it is weightless (see Ripa 1990). As noted by Ripa (1990), this fact can be used to deduce bounds on the phase velocity of unstable waves. In the multilayer quasi-geostrophic model, the definition of $\boldsymbol{L}$, and subsequent integration by parts, yield an explicit expression for a mode's pseudomomentum,

$$\{\boldsymbol{q}_a, \boldsymbol{L}^{-1}\boldsymbol{q}_a\} = -\mathrm{Re}\left(\{\boldsymbol{\chi}_{a,y}, \boldsymbol{V}\boldsymbol{\chi}_{a,y}\} + k^2\{\boldsymbol{\chi}_a, \boldsymbol{V}\boldsymbol{\chi}_a\} - \{\boldsymbol{\chi}_a, \boldsymbol{VKTK}\boldsymbol{\chi}_a\} - \beta\{\boldsymbol{\chi}_a, \boldsymbol{\chi}_a\}/2\right),$$

where $\boldsymbol{\chi}_a = \boldsymbol{V}^{-1}\boldsymbol{\psi}_a$, $\boldsymbol{V} = \boldsymbol{U} - c_a\boldsymbol{I}$. For an unstable mode, the vanishing of this quantity leads to an expression for the real part of the phase velocity

$$\mathrm{Re}(c_a) = \frac{\{\boldsymbol{\chi}_{a,y}, \boldsymbol{U}\boldsymbol{\chi}_{a,y}\} + k^2\{\boldsymbol{\chi}_a, \boldsymbol{U}\boldsymbol{\chi}_a\} - \{\boldsymbol{\chi}_a, \boldsymbol{UKTK}\boldsymbol{\chi}_a\} - \beta\{\boldsymbol{\chi}_a, \boldsymbol{\chi}_a\}/2}{\{\boldsymbol{\chi}_{a,y}, \boldsymbol{\chi}_{a,y}\} + k^2\{\boldsymbol{\chi}_a, \boldsymbol{\chi}_a\} - \{\boldsymbol{\chi}_a, \boldsymbol{KTK}\boldsymbol{\chi}_a\}}. \tag{3.4}$$

Now, the eigenvalues of $\boldsymbol{KTK}$ are non-positive (Liu Yongming & Mu Mu 1992) and the Poincaré inequality gives the bounds

$$\min_{i,y} U_i - \frac{\beta}{2[k^2 + (\pi/l)^2]} < \mathrm{Re}(c_a) < \max_{i,y} U_i$$

which are similar to those obtained for a vertically continuous model (e.g. Pedlosky 1987, §7.5).

Once the solutions of (3.2) are known, the solution of the evolution equations (3.1) is readily found: it can be expressed as the superposition of modes $\boldsymbol{q}_a$ with the time dependence $\exp(-\mathrm{i}\omega_a t)$ where $\omega_a = k_a c_a$. However, for particular k_m the situation can be complicated by the presence of marginally stable modes which correspond to the existence of multiple real eigenvalues with dependent eigenvectors. Here we consider only the case of a double root, say c_m, in the same manner as Romanova (1992). The corresponding general solution can be written

$$\boldsymbol{q}(y,t) = \boldsymbol{q}_{m_0}(y)\exp \mathrm{i}k_m(x-c_m t) - \mathrm{i}k_m \boldsymbol{q}_{m_1}(y)\, t \exp \mathrm{i}k_m(x-c_m t),$$

where $\boldsymbol{q}_{m_0}$ and $\boldsymbol{q}_{m_1}$ satisfy

$$\boldsymbol{N}_m \boldsymbol{q}_{m_1} = c_m \boldsymbol{L}^{-1}\boldsymbol{q}_{m_1}, \quad \boldsymbol{N}_m \boldsymbol{q}_{m_0} = \boldsymbol{L}^{-1}(c_m \boldsymbol{q}_{m_0} + \boldsymbol{q}_{m_1}). \tag{3.5a, b}$$

As $\boldsymbol{q}_{m_1}$ is the limit between stable and unstable modes, it is obvious that $\{\boldsymbol{q}_{m_1}, \boldsymbol{L}^{-1}\boldsymbol{q}_{m_1}\} = 0$, so that the non-homogeneous equation has a solution even though the operator $\boldsymbol{N}_m - c_m \boldsymbol{L}^{-1}$ is singular. This solution can easily be shown to have the form

$$\boldsymbol{q}_{m_0} = \boldsymbol{q}'_{m_1} + \mu \boldsymbol{q}_{m_1},$$

where the prime indicates the derivative with respect to c_m and where μ is an arbitrary constant. An appropriate choice of this constant leads to the orthogonality relation $\{\boldsymbol{q}_{m_0}, \boldsymbol{L}^{-1}\boldsymbol{q}_{m_0}\} = 0$. Thus the marginally stable modes have a single contribution to the pseudomomentum $S_m \equiv \{\boldsymbol{q}_{m_1}, \boldsymbol{L}^{-1}\boldsymbol{q}_{m_0}\}$.

The orthogonality relations deduced above are useful because they show that the pseudomomentum can be decomposed into independent contributions from each stable mode, each growing–decaying pair of modes, and each pair of modes corresponding to marginal stability. The same is true for the pseudoenergy and can be demonstrated using the orthogonality relations in the sense of pseudoenergy. These relations are easily found by rewriting the eigenvalue problem (3.1) as

$$\boldsymbol{R}_a \boldsymbol{q}_a = c_a \boldsymbol{N}_a \boldsymbol{q}_a$$

and by noting that $\boldsymbol{R}_a \equiv \boldsymbol{N}_a \boldsymbol{L} \boldsymbol{N}_a$ is Hermitian. It turns out that

$$\{\boldsymbol{q}_a, \boldsymbol{N}_b \boldsymbol{q}_b\} = \delta_{a^*,b}\tilde{E}_b \tag{3.6}$$

and that

$$\tilde{E}_b = c_b \tilde{P}_b.$$

It can be seen from (2.5*a*) that the contribution of an isolated stable mode to the pseudoenergy is just given by $E_s \equiv \tilde{E}_s$ while the contribution of a pair of growing decaying modes is $F_u \equiv \tilde{E}_u$. For modes corresponding to marginal stability, it is straightforward to show that the following two products do not vanish: $\{\boldsymbol{q}_{m_0}, \boldsymbol{N}_m \boldsymbol{q}_{m_0}\} = S_m$ and $G_m \equiv \{\boldsymbol{q}_{m_1}, \boldsymbol{N}_m \boldsymbol{q}_{m_0}\}$, this latter product being related to S_m through $G_m = c_m S_m$. The asymmetry between the orthogonality relations in the sense of pseudomomentum and of pseudoenergy for the marginally stable mode is merely a consequence of a particular choice of μ which favours the first kind of orthogonality.

A more general orthogonality relation can be deduced as

$$\{\boldsymbol{q}_a, (\boldsymbol{N}_b - \alpha \boldsymbol{L}^{-1})\boldsymbol{q}_b\} = \delta_{a^*,b}\tilde{E}^{\alpha}_b, \tag{3.7}$$

where $\tilde{E}^{\alpha}_b = \tilde{E}_b - \alpha \tilde{P}_b$ corresponds to the pseudoenergy in a frame of reference moving with velocity α (see VV). However, that relation becomes really useful when modes with $1/c_b = 0$ appear in the system. As it is not the case in the quasi-geostrophic models, we mainly use in the next sections the relation (3.3) but it should be kept in mind that (3.7) can be necessary for waves that are not baroclinic Rossby waves.

4. Interaction equations

We now expand the solution of the nonlinear equation (2.3) in a series of normal modes:

$$q(x,y,t) = \sum_{k_a}\sum_{n_a} A_a(t)\, q_a(y) \exp ik_a(x - c_a t). \tag{4.1}$$

Here, the sum on k_a has to be interpreted as an integral when the domain is unbounded in the x-direction. The sum on n_a is also an integral where the spectrum of the eigenvalues has continuous parts. As $\boldsymbol{N}_a$ does not depend on the sign of k_a, the mode a defined by (k_a, n_a, c_a, q_a) and the mode $-a$ defined by $(-k_a, n_a, (c_a)^*, (q_a)^*)$ are normal modes of the system representing the same physical state. The reality of the series (4.1) is therefore ensured provided that the modes a and $-a$ are always considered together, and that

$$A_{-a}(t) = [A_a(t)]^*.$$

The presence of $(c_a)^*$ and $(q_a)^*$ in the definition of mode $-a$ is required to ensure that mode $-a$ has the same temporal growth (or decay) as mode a when unstable. For clarity, it is useful to re-write (4.1) separating the stable, unstable and marginal modes:

$$\begin{aligned} q = {} & \sum_s A_s(t)\, q_s \exp ik_s(x - c_s t) \\ & + \sum_u [A_u(t)\, q_u \exp ik_u(x - c_u t) + A_{u^*}(t)(q_u)^* \exp ik_u(x - (c_u)^* t)] \\ & + \sum_m [A_{m_0}(t)\, q_{m_0} + A_{m_1}(t) q_{m_1}] \exp ik_m(x - c_m t). \end{aligned} \tag{4.2}$$

The amplitudes of the growing and decaying members of a pair of unstable modes are denoted A_u and A_{u^*}, respectively. There is no restriction on the relative values of these amplitudes and, in general, $A_{u^*} \neq A_u$. In the following the quadratic part of the pseudomomentum and of the pseudoenergy will prove crucial, although we deal with the fully nonlinear equations. Owing to the orthogonality relations (3.3) and (3.6), respectively, these quantities take the simple form

$$P^{(2)} = \tfrac{1}{2}\{\sum_s P_s |A_s|^2 + \sum_u [R_u (A_{u^*})^* A_u + \text{c.c.}] + \sum_m S_m [(A_{m_1})^* A_{m_0} + \text{c.c.}]\}, \tag{4.3a}$$

$$E^{(2)} = \tfrac{1}{2}\{\sum_s E_s |A_s|^2 + \sum_u [F_u (A_{u^*})^* A_u + \text{c.c.}] + \sum_m [G_m (A_{m_1})^* A_{m_0} + \text{c.c.} + S_m |A_{m_0}|^2]\}. \tag{4.3b}$$

The sums contain only one term for each growing–decaying pair of unstable modes or for each pair of marginally stable modes. Notice that the factor $\tfrac{1}{2}$ can be removed if one considers the sum only on distinct physical states, i.e. if the sum comprises only one term for each pair a and $-a$.

For zero-circulation disturbances, the expansion (4.1) or (4.2) is complete and the original system of partial differential equations (2.3) can be replaced by an infinite set of ordinary differential equations for the amplitudes $A_a(t)$. Following VV, these equations are obtained by introducing the expansion (4.2) into (2.3) and then by using the orthogonality relations (3.3) as projection operator. For a mode a, either stable or unstable, one obtains the equation

$$\dot{A}_a = \frac{\mathrm{i}}{2}\sum_{b,c} I_a^{bc} A_b^* A_c^* \exp(\mathrm{i}\Omega_{abc} t), \tag{4.4a}$$

where $\Omega_{abc} = \omega_a + (\omega_b)^* + (\omega_c)^*$ is a detuning parameter, and where the interaction coefficient is

$$I_a^{bc} = [\{\boldsymbol{q}_a^*, \boldsymbol{L}^{-1}\boldsymbol{S}_{bc}(\boldsymbol{\psi}_b^*, \boldsymbol{q}_c^*)\} + b \leftrightarrow c]/\tilde{P}_a. \tag{4.4b}$$

Here $\boldsymbol{S}_{bc}$ designates the vector $\boldsymbol{S}$ defined in (2.3) where the Jacobians become $J(\psi_b, q_c) = k_b \psi_b \partial_y q_c - k_c q_c \partial_y \psi_c$. All modes satisfying the interaction condition

$$k_a + k_b + k_c = 0 \tag{4.5}$$

have to be taken into account in the right-hand side of (4.4). Notice that (4.4) implies the assumption that $\tilde{P}_a$ does not vanish and therefore does not apply to marginally stable modes. For a marginally stable mode m, the projection on $\boldsymbol{q}_{m_0}$ and on $\boldsymbol{q}_{m_1}$ leads to a system of coupled equations for the corresponding amplitudes

$$\dot{A}_{m_0} = \frac{\mathrm{i}}{2}\sum_{b,c} I_{m_0}^{bc} A_b^* A_c^* \exp(\mathrm{i}\Omega_{mbc} t), \tag{4.6a}$$

$$\dot{A}_{m_1} + \mathrm{i}k_m A_{m_0} = \frac{\mathrm{i}}{2}\sum_{b,c} I_{m_1}^{bc} A_b^* A_c^* \exp(\mathrm{i}\Omega_{mbc} t), \tag{4.6b}$$

where the interaction coefficients are

$$\left.\begin{aligned} I_{m_0}^{bc} &= [\{\boldsymbol{q}_{m_1}, \boldsymbol{L}^{-1}\boldsymbol{S}_{bc}(\boldsymbol{\psi}_b^*, \boldsymbol{q}_c^*)\} + b \leftrightarrow c]/S_m \\ \text{and} \qquad I_{m_1}^{bc} &= [\{\boldsymbol{q}_{m_0}, \boldsymbol{L}^{-1}\boldsymbol{S}_{bc}(\boldsymbol{\psi}_b^*, \boldsymbol{q}_c^*)\} + b \leftrightarrow c]/S_m. \end{aligned}\right\} \tag{4.7}$$

The analogue of the linear part of equations (4.4*a*) and (4.6) has been derived by Romanova (1992) for a multilayer model of gravity waves. In a recent paper, the same author treats the problem arising for modes in the vicinity of marginal stability (Romanova 1994). For such a mode, the pseudomomentum is infinitesimal and the interaction coefficient tends to infinity. Therefore, the hypothesis of weakly nonlinear interaction breaks down, even for infinitesimal amplitudes of the modes, and the set of equations (4.4*a*) cannot be easily truncated. From the physical point of view, this comes from the fact that the nonlinear interaction of a mode near marginal stability cannot be treated without considering its companion mode with which it coalesces at the marginal point. Indeed, both modes have the same wavenumber and nearly the same frequency. Thus, when one mode is involved in a triad, its companion mode also interacts significantly with the other two members of the triad. To solve this problem, Romanova's basic idea is the introduction of new independent vectors, which are a linear combination of the eigenvectors corresponding to companion modes, in the vicinity of the marginal stability. The new vectors are chosen so that they correspond to $\boldsymbol{q}_{m_1}$ and $\boldsymbol{q}_{m_0}$ in the marginal stability limit.

5. Interaction properties and explosive resonant interaction

The conservation laws of pseudomomentum and pseudoenergy can be used to deduce relations between the three interaction coefficients of a wave triad, in a similar manner to that of Ripa (1983) for equatorial waves in a resting fluid. The present development extends the results of VV which considered only stable modes. It is worth emphasizing that the results do not depend on the exact nature of the physical system studied: they are directly related to the quadratic order of the nonlinearity, and to the existence of two exact nonlinear invariants. In particular, they apply to the Boussinesq model of gravity waves studied by VV. For simplicity, we extend the notation u and u^* (used in the previous section to designate conjugate unstable modes) to stable and marginally stable modes. By definition, s and s^* represent the same stable mode and m and m^* represent the two modes of marginal stability m_0 and m_1. The quadratic part of the pseudomomentum (4.3a) then takes the condensed form

$$P^{(2)} = \tfrac{1}{2}\sum_a \tilde{P}_a (A_{a^*})^* A_a. \tag{5.1}$$

Here, and contrary to (4.3), the sum on a runs over each individual mode, i.e. $a = u$, $a = u^*$ and $a = m_0$, $a = m_1$ have to be considered. It is always possible to define a coefficient τ_{abc} invariant under permutation of the indices so as to write the cubic part of the pseudomomentum as

$$P^{(3)} = \tfrac{1}{6}\sum_{abc} \tau_{abc} A_a A_b A_c \exp(-\mathrm{i}\tilde{\Omega}_{abc} t), \tag{5.2}$$

where $\tilde{\Omega}_{abc} = \omega_a + \omega_b + \omega_c$. Starting from (5.1) and using (4.4a) and (4.6) gives the general expression (including the marginal modes) for the variation of $P^{(2)}$:

$$\dot{P}^{(2)} = \tfrac{1}{2}\sum_{abc} \mathrm{Im}\,\{\tilde{P}_a (I^{bc}_{a^*})^* A_a A_b A_c \exp[-\mathrm{i}(\Omega_{a^*bc})^* t]\},$$

while the variation of $P^{(3)}$ reads at leading order

$$\dot{P}^{(3)} = \tfrac{1}{3}\sum_{abc} \mathrm{Im}\,[\tilde{\Omega}_{abc} A_a A_b A_c \tau_{abc} \exp(-\mathrm{i}\tilde{\Omega}_{abc} t)] + O(A^4).$$

Conservation of the pseudomomentum requires $\dot{P}^{(2)} + \dot{P}^{(3)} = O(A^4)$ and leads thus to the relation between the interaction coefficients

$$\tilde{P}_{a^*} I^{bc}_{a^*} + \tilde{P}_{b^*} I^{ca}_{b^*} + \tilde{P}_{c^*} I^{ab}_{c^*} = -(\tilde{\Omega}_{abc}\tau_{abc})^*, \tag{5.3a}$$

where the equality $\tilde{\Omega}_{abc} = (\Omega_{a^*bc})^*$ has been used. A similar treatment of the pseudoenergy conservation leads to

$$c_{a^*}\tilde{P}_{a^*} I^{bc}_{a^*} + c_{b^*}\tilde{P}_{b^*} I^{ca}_{b^*} + c_{c^*}\tilde{P}_{c^*} I^{ab}_{c^*} = -(\tilde{\Omega}_{abc} \upsilon_{abc})^*, \tag{5.3b}$$

where υ_{abc} defines the cubic part of the pseudoenergy and where (3.6) has been used. Contrary to (5.3a), (5.3 b) is not general; if a mode, say a, is a marginal mode m_0, the substitution

$$c_{a^*}\tilde{P}_{a^*} I^{bc}_{a^*} \leftarrow c_m S_m I^{bc}_{m_1} + S_m I^{bc}_{m_0}$$

must be made. Notice that the interaction coefficients involved in (5.3) are not exactly those appearing in the three evolution equations of the modes a, b and c. They may however be related to them: for instance, if a is an unstable mode while b and c are stable, one can easily show that $I^{bc}_{a^*} = (I^{bc}_a)^*$, $I^{ca}_{b^*} = I^{ca}_b$ and $I^{ab}_{c^*} = I^{ab}_c$.

The relations (5.3) are not useful in general since the coefficients τ_{abc} and υ_{abc} are difficult to evaluate: they may be deduced from the finite-amplitude expressions for the pseudomomentum and pseudoenergy (2.5) but this requires long calculations (see VV for an example). From a theoretical point of view, however, (5.3) can be used to obtain an expression for the cubic part of the invariants, starting from their quadratic part (VV). The most interesting conclusions that can be drawn from (5.3) occur when their right-hand sides vanish. In particular, this is the case when there is no basic flow or when the basic velocity is constant in each layer (generalized Phillips models). Indeed the pseudomomentum and pseudoenergy are then exactly quadratic, whence $\tau_{abc} = v_{abc} = 0$. Another possibility is that the three modes satisfy the resonance condition

$$\omega_a + \omega_b + \omega_c = 0, \tag{5.4}$$

so that $\tilde{\Omega}_{abc} = 0$. Then, together with (4.5) and (5.4), (5.3*a*) and (5.3*b*) yield the factorization relation

$$\frac{\tilde{P}_{a*} I^{bc}_{a*}}{k_a} = \frac{\tilde{P}_{b*} I^{ca}_{b*}}{k_b} = \frac{\tilde{P}_{c*} I^{ab}_{c*}}{k_c}, \tag{5.5a}$$

which shows that the interaction inside a resonant triad can be described using only one interaction coefficient. Classically, the factorization relation found involves the pseudoenergy rather than the pseudomomentum and takes the form

$$\frac{\tilde{E}_{a*} I^{bc}_{a*}}{\omega_a} = \frac{\tilde{E}_{b*} I^{ca}_{b*}}{\omega_b} = \frac{\tilde{E}_{c*} I^{ab}_{c*}}{\omega_c}. \tag{5.5b}$$

The latter expression may be fruitfully formulated in a moving reference frame, the pseudoenergies, as well as the frequencies, being frame-dependent.

In what follows, we concentrate on the baroclinic Rossby waves, i.e. the linearly stable modes. A wave can be considered as a basic flow whose stability may be investigated and (5.5) can be used to extend Hasselmann's criterion on wave instability through resonant interaction (Hasselmann 1967). Consider a wave *a* which is disturbed by two stable modes *b* and *c* such that (4.5) and (5.4) are fulfilled. The evolution of the disturbing waves is studied by assuming that $|A_a| \gg |A_b|, |A_c|$, so that $|A_a|$ can be taken constant. Solving the evolution equations for A_b and A_c shows that they grow exponentially for any amplitude A_a provided that (e.g. Craik 1985)

$$I^{ca}_b I^{ab}_c > 0. \tag{5.6}$$

If this condition is satisfied, the wave *a* is unstable through nonlinear interaction with the modes *b* and *c*. From (5.5*a*) and (5.5*b*), condition (5.6) can be re-formulated:

$$\frac{P_b k_c}{P_c k_b} > 0 \quad \text{or} \quad \frac{E_b \omega_c}{E_c \omega_b} > 0.$$

The analogues of Arnol'd's theorems demonstrating the stability of the basic flow are often based on the sign-definiteness of the pseudomomentum or of the pseudoenergy (e.g. Ripa 1992). If this sign-definiteness can be proved (as it is obviously the case when there is no basic flow), the above expressions can be simplified to give useful criteria for the instability of the wave *a* through resonant interaction with *b* and *c*: if the pseudomomentum is sign-definite then instability of *a* occurs only if $|k_a| > |k_b|, |k_c|$, while if the pseudoenergy is sign-definite then instability occurs only if $|\omega_a| > |\omega_b|, |\omega_c|$. This latter condition corresponds to Hasselmann's criterion for a resting fluid. In the context of a multilayer quasi-geostrophic model, it has only been checked numerically for the two-layer model by Jones (1979*a*). The demonstration given here clearly illustrates that this criterion is a general consequence of the conservation laws. Furthermore, it is simple to recover and extend Jones' conclusion about the direction of energy transfer between waves and by noting that $k_a/(c_b - c_c)$ is invariant under circular permutation of the indices: the wave with the largest wavenumber possesses the intermediate phase velocity and hence the intermediate pseudowavenumber (given by $(k^2 + R^{-2})^{1/2}$, where R is the mode deformation radius; see Jones 1979*a*). More general stability conditions for parallel flow do not require the pseudomomentum or the pseudoenergy to be independently sign-definite: condition (2.6) ensures that the combination $E-\alpha P$ is positive definite, while (2.7) ensures that it is negative definite, at least for the zero-circulation disturbances we are dealing with (Mu Mu *et al.* 1994). If one of those criteria holds for a particular α, straightforward manipulation shows that a criterion for the instability of wave a can be written as

$$(\alpha k_b-\omega_b)(\alpha k_c-\omega_c) > 0, \tag{5.7}$$

stating that wave a has the largest apparent frequency in the frame moving with velocity α. The criterion involving only the wavenumbers (frequencies) is recovered when (2.6) or (2.7) hold with $\alpha = \infty$ (0).

The important property of triad interactions in shear flow that constitutes the main subject of the present paper is the existence of the explosive resonant interaction (ERI) of baroclinic Rossby waves. This phenomenon, corresponding to the simultaneous growth of the three waves in the triad, leads to finite-time singularity in the wave amplitudes, and is a nonlinear instability mechanism for the basic parallel flow. It has been found by Cairns (1979) and Craik & Adam (1979) (see also Craik 1985) in the case of interacting interfacial gravity waves, and has been investigated by Romanova (1987) for a three-layer Phillips model. Here we discuss its general properties for the N-layer model before considering more particularly the generalized Phillips models in the next section. Consider three linearly stable waves a, b and c satisfying (4.5) and (5.4), whose evolution is governed by a system of three equations like (4.4a), coupled by the nonlinear terms. ERI occurs if the three interaction coefficients of these equations have the same sign. As can be seen from (5.5a), this is equivalent to the requirement that P_a/k_a, P_b/k_b and P_c/k_c have the same sign, while the analogous condition derived from (5.5b) concerns the sign of E_a/ω_a, E_b/ω_b and E_c/ω_c. Taking into account the interaction condition (4.5) and the resonance condition (5.4), two equivalent necessary conditions for ERI may be stated: ERI occurs if the wave with the largest wavenumber (frequency) has opposite-signed pseudomomentum (pseudoenergy) from the other two waves. Evidently, the fulfilment of the stability conditions (2.6) or (2.7) must preclude ERI. This is obvious when the pseudomomentum or the pseudoenergy is sign-definite but it seems worth showing when (2.6) or (2.7) hold with $\alpha \neq 0, \infty$. If I_b^{ca} and I_c^{ab} have the same sign, then (5.7) is satisfied and it is easy to prove, using (4.5), (5.4) and the sign-definiteness of $E-\alpha P$ that

$$(\alpha k_a-\omega_a)(\alpha k_b-\omega_b) < 0,$$

and thence that I_a^{bc} and I_b^{ca} are oppositely signed. The discussion above bears on the sign of the pseudomomentum or pseudoenergy of modes, which only involves the quadratic part of their general expression (2.5). This highlights the general statement that formal stability (i.e. sign-definiteness of the quadratic part of pseudomomentum or pseudoenergy) is sufficient to prevent ERI. In quasi-geostrophic models, however, conditions for formal and nonlinear stability are the same for disturbances with vanishing circulation.

An interesting point may be noted by multiplying (3.4) by c_a to obtain the expression for wave pseudoenergy:

$$E_a = -\mathrm{Re}\left(\{\chi_{a,y}, c_a \boldsymbol{V}\chi_{a,y}\}+k^2\{\chi_a, c_a \boldsymbol{V}\chi_a\}-\{\chi_a, c_a \boldsymbol{VKTK}\chi_a\}-\beta c_a\{\chi_a,\chi_a\}/2\right).$$

Following Becker & Grimshaw (1993), the invariance in translation of the condition for ERI can be used: in a frame such that $U_m \equiv \min_{i,y} U_i < 0 < U_M \equiv \max_{i,y} U_i$, $c_a \boldsymbol{V}$ is negative definite if $c_a < U_m$ or $c_a > U_M$. In Becker & Grimshaw's model involving gravity waves, this condition ensures that the pseudoenergy of the modes is always positive, and hence that ERI is impossible. ERI requires then that $U_m < c_a < U_M$ for at least one member of the triad, i.e. the existence of a critical level in the case of a continuous model. Here, owing to the β-effect, such a conclusion cannot be drawn *a priori* as the pseudoenergy is not sign-definite, even if c_a is outside the range of U.

The classical graphical method for the study of resonant triads using the dispersion curves (see Craik & Adam 1979; Tsutahara 1984) can be combined with the criterion above to locate explosively resonant triads in a simple manner: the redrawn dispersion curve and the original one along which its origin is moving must have the same sign of pseudomomentum, while the second original curve which is intersected by the redrawn curve (and which automatically has the largest wavenumber) must have an oppositely signed pseudomomentum. This procedure has proved very useful to get a first estimate of the regions where ERI is possible.

Before turning to the study of generalized Phillips models, a remark is necessary about the modes belonging to the continuous spectrum that may exist when the basic flow is sheared in the y-direction. As mentioned in §3, these modes are singular at a critical level and, hence, the interaction coefficients (4.4*b*) are not regular functions of the continuous index. However, all the relations presented above are rigorous if the interaction coefficients are interpreted as distributions. This means that they have to be considered under an integral on the continuous index n_a or, equivalently, on the corresponding phase velocity c_a (see VV). Physically, this means that interaction between isolated modes of the continuous spectrum makes no sense whereas the interaction between packets of such modes may be analysed. Critical level singularity is indeed known to be an artificial difficulty raised by the spectral decomposition rather than a real physical property, at least for initial value problems (see Tung 1983).

6. Generalized Phillips models

6.1. *N-layer model*

Generalized Phillips models are constructed from the model described in §2 by assuming that the basic velocity does not depend on y. The matrices $\boldsymbol{U}$ and $\boldsymbol{L}$ are therefore constant and the solutions of the eigenvalue problem (3.2) are simply given by

$$q_a(y) = \hat{q}_a \sin(n_a \pi y/l),$$

for $n_a = 1, 2, \ldots$. For simplicity we will ignore the ˆ and consider the modes corresponding the same wavenumber k and index n. The index a is thus used to distinguish the N solutions of the algebraic eigenvalue problem

$$\boldsymbol{N}_K \boldsymbol{q}_a = c_a \boldsymbol{L}^{-1} \boldsymbol{q}_a. \tag{6.1}$$

Here $\boldsymbol{N}_K$ is simply the matrix deduced from $\boldsymbol{N}$ where the substitution $\nabla^2 \leftarrow -K \equiv -(k^2 + n^2\pi^2/l^2)$ is made. This formulation emphasizes that (6.1) only depends on K and not on k and n individually. The phase velocity c_a is a solution of the dispersion relation

$$D(c, K) \equiv \det[\boldsymbol{M}_K(c)] = 0, \tag{6.2}$$

where we have introduced the matrix $\boldsymbol{M}_K(c) \equiv \boldsymbol{N}_K - c\boldsymbol{L}^{-1}$. Many different forms of the dispersion relation can be found, depending on the formulation of the eigenvalue problem. In general, $D(c, K)$ can be multiplied by any non-vanishing function $f(c)$ without altering the solutions c_a. Advantage may be taken of this freedom to choose a dispersion relation linked to the physical properties of the system. In particular, it is often chosen so that its derivative with respect to the frequency is directly related to the mode (pseudo)energy (see Cairns 1979). In her study of the three-layer Phillips model, Romanova (1987) proposed a dispersion relation whose derivative can be interpreted as the pseudomomentum of the mode but her formulation rests upon peculiar algebraic manipulation and cannot be easily extended. In the context of a non-rotating N-layer model, the same author has recently given a rigorous explanation of the link between the dispersion relation and the invariant of the linear system (Romanova 1992). Using the same type of arguments, we argue that a new dispersion relation, whose derivative with respect to c gives the pseudomomentum, can be constructed from (6.1). The derivation is similar to Romanova's although the physical system (and hence the matrix $\boldsymbol{M}_K$) is very different. Consider the new dispersion relation

$$\tilde{D}(c,K) \equiv D(c,K) / \sum_{i=1}^{N} [M_K(c)]_i, \tag{6.3}$$

where $[M_K]_i$ designates the ith principal minor of $\boldsymbol{M}_K$ i.e. the minor of its ith diagonal element. The derivative of (6.3) with respect to c, taken for $c = c_a$, is

$$\left.\frac{\partial \tilde{D}}{\partial c}\right|_{ca} = \left\{\sum_{i=1}^{N} [M_K(c_a)]_i\right\}^{-1} \left.\frac{\partial D}{\partial c}\right|_{c_a}. \tag{6.4}$$

Noting that
$$D(c,K) = \left[\prod_{a=1}^{N} (c-c_a)\right]\left(\prod_{j=1}^{N} Q_{jy}\right)^{-1},$$

and introducing the matrix $\tilde{\boldsymbol{M}}_K = -\boldsymbol{L}\boldsymbol{M}_K$, we can write

$$\frac{\partial D}{\partial c} = \left\{\sum_{i=1}^{N} [\tilde{M}_K(c)]_i\right\}\left(\prod_{j=1}^{N} Q_{jy}\right)^{-1}. \tag{6.5}$$

To deduce the latter equality, we have used the fact that the first factor of the right-hand side is an invariant of $\tilde{\boldsymbol{M}}_K$ and can thus be calculated in diagonal form. The minors of $\tilde{\boldsymbol{M}}_K$ and of $\boldsymbol{M}_K$ are related through

$$[\tilde{M}_K(c)]_i = [M_K(c)]_i \left(\prod_{\substack{j=1\\ j\neq i}}^{N} Q_{jy}\right)$$

so that (6.5) becomes

$$\frac{\partial D}{\partial c} = \sum_i Q_{iy}^{-1} [M_K(c)]_i. \tag{6.6}$$

When $c = c_a$, $\boldsymbol{M}_K(c_a)\boldsymbol{q}_a = 0$ and one can show (see the Appendix) that

$$[M_K(c_a)]_i \, q_{a,1}^2 = [M_K(c_a)]_1 \, q_{a,i}^2. \tag{6.7}$$

Using this property in (6.4) and (6.6) we write the former equation as

$$\left.\frac{\partial \tilde{D}}{\partial c}\right|_{c_a} = \left(\sum_{i=1}^{N} q_{a,i}^2\right)^{-1}\left(\sum_{i=1}^{N} Q_{iy} q_{a,i}^2\right) = \{\boldsymbol{q}_a^*, \boldsymbol{q}_a\}^{-1}\{\boldsymbol{q}_a^*, \boldsymbol{L}^{-1}\boldsymbol{q}_a\},$$

or, equivalently,
$$\left.\frac{\partial \tilde{D}}{\partial c}\right|_{c_a} = \{\boldsymbol{q}_a^*, \boldsymbol{q}_a\}^{-1} \tilde{P}_a. \tag{6.8}$$

As desired, the derivative of the new dispersion relation is directly related to the mode pseudomomentum. For a wave, it is equal to the ratio of the pseudomomentum to the potential enstrophy. This latter quantity being positive definite (and also a good indicator of the physical importance of the mode), the sign of the pseudomomentum, essential for the ERI, is directly deduced from the dispersion relation. Compared to Romanova (1992), we have introduced the denominator in (6.3) as the sum of the principal minors while she only used a single minor. We argue that this choice renders unambiguous the sign of the pseudomomentum, at least for waves.

We now turn to a discussion of the stability conditions and their relation with the linear problem and the ERI. First, notice that conditions (2.6) and (2.7) are very different: the former depends only on the basic velocity field, while the latter depends also on the channel geometry through K_0. An equivalent condition to (2.6) can be stated as the search for a layer j such that

$$Q_{iy}^{-1}(U_i - U_j) < 0, \quad \forall i \neq j. \tag{6.9}$$

The equivalence is readily demonstrated: if (6.9) is satisfied, then (2.6) is also satisfied with $\alpha = U_j + \delta Q_{jy}$ where δ is an arbitrarily small positive constant. Conversely, (2.6) may be written

$$U_m < \alpha < U_l,$$

where the layer indices m and l are such that $Q_{my} > 0$ and $Q_{ly} < 0$. Taking the layer j corresponding to the velocity $\max_m U_m$ or $\min_l U_l$, we directly deduce (6.9).

If (6.9) is not satisfied, linear stability is examined by checking whether there exists modes with non-zero imaginary part of c. The linearized quasi-geostrophic models manifest a short-wave cut-off, meaning that all modes are stable for $K > K_{cr}(U_i, F_i)$. Thus, if

$$K_0 \equiv (\pi/l)^2 > K_{cr}(U_i, F_i), \tag{6.10}$$

the system is linearly stable for any disturbance. We shall now prove that if, in addition, (2.7*a*) is satisfied, then (2.7*b*) automatically follows, and therefore the system is also nonlinearly stable. Notice that, as for (2.6) and (6.9), (2.7*a*) is equivalent to

$$Q_{iy}^{-1}(U_i - U_j) > 0, \quad \forall i \neq j, \tag{6.11}$$

for a particular j. To show nonlinear stability, it must first be recognized that the optimum choice of γ_i in (2.7*a*) for the generalized Phillips models is

$$\gamma_i = Q_{iy}^{-1}(U_i - \alpha) > 0, \quad i = 1, N. \tag{6.12}$$

From (2.7*b*), (3.1) and (6.2), we find that the matrix which has to be positive definite is simply $-\boldsymbol{M}_{K_0}(\alpha)$. Now, taking into account (6.1), the matrix $\boldsymbol{N}_{K_0}$ can be expanded in the basis of the vector $\boldsymbol{L}^{-1}\boldsymbol{q}_a$ following

$$\boldsymbol{N}_{K_0} = \sum_{a=1}^{N} \frac{c_a}{P_a} \boldsymbol{L}^{-1}\boldsymbol{q}_a \otimes \boldsymbol{L}^{-1}\boldsymbol{q}_a.$$

Here c_a and $\boldsymbol{q}_a$ are the N eigenvectors and eigenvalues of (6.1) for $K = K_0$; they represent stable waves as we have assumed (6.10). For any real vector $\boldsymbol{x}$, it is easy to show that

$$\boldsymbol{x}^T \boldsymbol{M}_{K_0}(\alpha)\, \boldsymbol{x} = \sum_{a=1}^{N} P_a(c_a - \alpha)\, x_a^2,$$

where x_a is the coefficient of the expansion of $\boldsymbol{x}$ in the $\boldsymbol{q}_a$ basis. The nonlinear stability condition (2.7*b*) is intimately connected with the linear eigenvalue problem; indeed, it is equivalent to finding an α such that

$$P_a(c_a - \alpha) < 0, \quad a = 1, N. \tag{6.13}$$

This latter condition may be checked when the dispersion relation (6.3) is known. To prove that (6.11) leads to (6.13), we use the following argument: in the whole domain where $K_0 > K_{cr}$, the 'topology' of the eigenvalues and eigenvectors does not change, in the sense that the ordering of the c_a and the sign of the P_a do not change. Indeed, such changes only occur when two modes coalesce, namely when the limit K_{cr} is crossed. Each mode may thus be characterized by its limit when $K_0 \to \infty$, i.e. when $l \to 0$. In this limit, the linear coupling between adjacent layers disappears and each mode a corresponds to a single layer i with

$$c_i \to U_i, \quad \boldsymbol{q}_i \to \boldsymbol{e}_i, \quad P_i \to -Q_{iy}^{-1}, \tag{6.14}$$

where $\boldsymbol{e}_i$ is the basis unit vector. For finite $K_0 > K_{cr}$, (6.13) transforms into

$$Q_{iy}^{-1}(c_i - \alpha) > 0, \quad i = 1, N. \tag{6.15}$$

Here, c_i refers to the phase velocity that tends to U_i, and is such that

$$c_i = U_i - \epsilon_i Q_{iy},$$

where $\epsilon_i > 0$ because $\mathrm{d}c_i/\mathrm{d}K$ may only change sign for $K_0 \leqslant K_{cr}$. For moderate value of K_0, ϵ_i may be of order unity. If (6.11) is satisfied for a particular j, let us choose

$$\alpha = c_j - \delta Q_{jy}, \quad \text{with} \quad 0 < \delta \ll 1.$$

Evidently, (6.12) and (6.15) are satisfied for $i = j$. This is also the case for $i \neq j$ since

$$Q_{iy}^{-1}(U_i - \alpha) = \epsilon_i + Q_{iy}^{-1}(c_i - c_j) + \delta Q_{iy}^{-1} Q_{jy} > 0$$

and

$$Q_{iy}^{-1}(c_i - \alpha) = Q_{iy}^{-1}(c_i - c_j) + \delta Q_{iy}^{-1} Q_{jy} > 0.$$

The two inequalities hold for sufficiently small δ because $c_i - c_j$ has the same sign as $U_i - U_j$.

When neither (6.9) nor (6.11) can be satisfied, nonlinear stability is not proved, even in the linearly stable domain defined by (6.10). As discussed in §6.2, this situation never occurs in the two-layer model, so that we may assume $N \geqslant 3$. We now show that the system is actually unstable and that ERI provides the necessary nonlinear mechanism of instability. In fact, three layers a, b and c then exist with

$$Q_{ay}^{-1}(U_a - U_c)\, Q_{by}^{-1}(U_b - U_c) < 0. \tag{6.16}$$

Under the above hypothesis, Q_{ay} and Q_{by} must have the same sign; otherwise, it is easy to show that (6.9) or (6.11) hold with $j = a$ or b instead of c. By the same argument, Q_{cy} has the opposite sign to Q_{ay} and Q_{by}. We consider a resonant triad of waves such that $K > K_0$ and $K \gg 1$, which correspond to the three layers a, b and c. From the interaction and resonance conditions (4.5) and (5.4), and taking (6.14) into account, we obtain

$$k_c = -\frac{U_a - U_b}{U_c - U_b} k_a = -\frac{U_b - U_a}{U_c - U_a} k_b.$$

Together with (6.16), these relations show that k_c is oppositely signed to k_a and k_b, i.e. that wave c is the largest-wavenumber member of the triad. From (6.14), it is also clear that the pseudomomentum of c is oppositely signed to the pseudomomentum of a and b. Using the criterion of §5, we conclude that ERI occurs in the triad a, b, c.

In summary, we have shown that a generalized Phillips model is stable if (6.9) is satisfied or if (6.10) and (6.11) are satisfied. When this is not the case, the model is

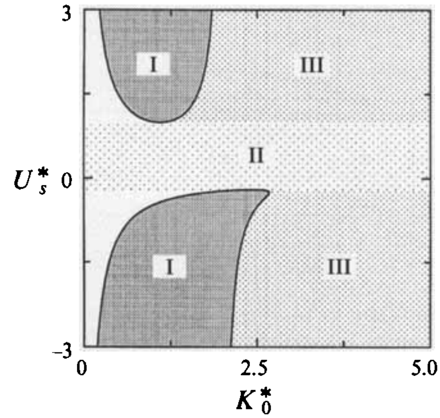

FIGURE 1. Stability diagram of the two-layer Phillips model for $F_2 = 0.2F_1$. The parameters are $K_0^* = K_0/(F_1 F_2)^{1/2}$ and $U_s^* = F_2(U_1 - U_2)/\beta$. Region I is linearly unstable for disturbance of wavenumber K_0 and the unshaded region is linearly unstable for smaller-scale disturbances. Regions II and III are proved nonlinearly stable by (6.9) and (6.11), respectively.

unstable whether by linear instability (if (6.10) is violated) or by ERI. Criteria (6.9) and (6.10) or equivalently, the original formulation (2.6) and (2.7) due to Ripa (1992, 1993) and Mu Mu *et al.* (1994), are thus necessary and sufficient conditions for nonlinear stability for the generalized Phillips model. Our developments show that (2.7*b*) is a direct consequence of (2.7*a*) (or (6.11)) when the model is linearly stable, i.e. when (6.10) is fulfilled. They also reveal the importance of the ERI of baroclinic Rossby waves: this rather simple mechanism suffices to explain all the regions in the parameter space where the model is unstable, albeit classed as stable by a linear (normal modes) analysis.

6.2. *Two-layer model*

The two-layer model is the original model whose linear stability was investigated by Phillips (1954). Ripa (1992, 1993) and Mu Mu *et al.* (1994) have studied its nonlinear stability in the case $\beta = 0$, and in the general case, respectively. They conclude that linear and nonlinear stability criteria coincide, i.e. that the model is nonlinearly stable as soon as it is linearly stable. This result is immediately recovered using our preceding criteria: once (6.10) is assumed, one of the conditions (6.9) or (6.11) is necessarily satisfied, since they are complementary. It can be noted that the above-mentioned authors' demonstration requires the extensive resolution of the linear instability problem, as well as the explicit formulation of (2.7*b*). Here, on the other hand, the same conclusion is obtained through a simple inspection of the basic velocity and potential vorticity gradient fields.

The stability conditions can be visualized on the classical picture of the marginal stability curve (e.g. Pedlosky 1987, §7.11) by taking the minimum wavenumber K_0 rather than a particular wavenumber K as the abscissa. A given model, defined by its basic velocity field and its geometry, is then represented by a characteristic point on the plane (K_0, $U_s \equiv U_1 - U_2$), while the stability of a particular mode disturbing the model depends on the location of the point obtained by translating this characteristic point by

$$\Delta K = K - K_0 = k^2 + (n^2 - 1)(\pi/l)^2 > 0.$$

A system is thus linearly stable when there is no linear instability domain to the right of its characteristic point. Figure 1 displays the stability diagram in the case $F_1 = 0.2F_2$.

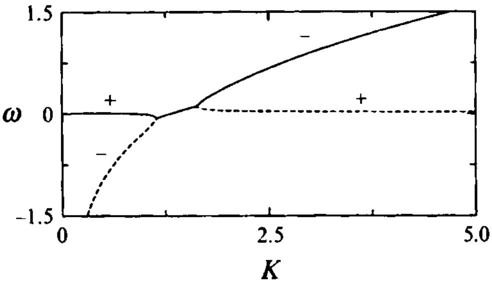

FIGURE 2. Dispersion relation of the two-layer Phillips model for $F_1 = F_2 = 1$, $\beta = 1$, $U_1 = 1.07$ and $U_2 = 0$. The sign of the pseudomomentum of each mode is indicated.

Region I is the usual linear instability region and the unshaded region to its left also corresponds to linearly unstable models in which only modes with sufficiently large ΔK grow. Regions II and III are nonlinearly stable by (6.9) and (6.11), respectively. It is clear from the figure that there is no possibility for a nonlinear instability where the model is linearly stable.

Romanova (1987) has claimed that the two-layer model cannot experience ERI, arguing that the sum of two waves with the same pseudoenergy sign could not be a wave with oppositely signed pseudoenergy when the three waves constituted a resonant triad. However, her argument holds only if the three waves lie on the same side of the marginal stability curve, i.e. if their representative points are all located either in the unshaded region or in the regions II and III. Thanks to the graphical method, it is clear from figure 2, which displays a typical dispersion curve with the pseudomomentum signs, that ERI is possible provided that one wave is on one side of the unstable domain (where both frequencies are conjugate) and the other two waves are on the other side. Thus ERI can occur, but only when linear instability occurs too. The parameters corresponding to this figure are

$$F_1 = F_2 = 1, \quad \beta = 1, \quad U_1 = 1.07, \quad U_2 = 0,$$

and the flow is slightly supercritical. We have numerically searched for triads undergoing ERI in these conditions. The three waves a, b and c are defined by their meridional indices

$$n_a = 0.2l, \quad n_b = 0.6l, \quad n_c = 0.8l,$$

and by their branch on the dispersion curve shown in figure 2: a belongs to the dashed branch with $P_a < 0$, b belongs to the solid branch with $P_b < 0$ and c belongs to the dotted branch with $P_c > 0$. In a resonant triad, the wavenumber k_c always has the largest magnitude; it can be considered negative so that k_a and k_b are positive. Figure 3 presents the wavenumbers k_b and $-k_c$ and the interaction coefficients I_a^{bc}, I_b^{ca} and I_c^{ab} as a function of the wavenumber k_a. The normal modes have been normalized so that the potential enstrophy of each mode is equal to unity. It can be seen from the figure that the three interaction coefficients rapidly vary with the triad wavenumbers but that they all have the same order. For $k_a \to 0.86$, wave a tends to marginal stability and an interaction coefficient tends to infinity, as expected. In this region, the new variables introduced by Romanova (1994) have to be used to investigate the behaviour of the interacting waves. The cusp seen on the three interaction coefficients curves for $k_a =$ 0.13 is actually due to a local vanishing of the coefficients. It is obvious from (5.5) that the three coefficients have to vanish at the same time. Note that during the calculation, we have used (5.5) as a check of the numerical exactness of our resolution.

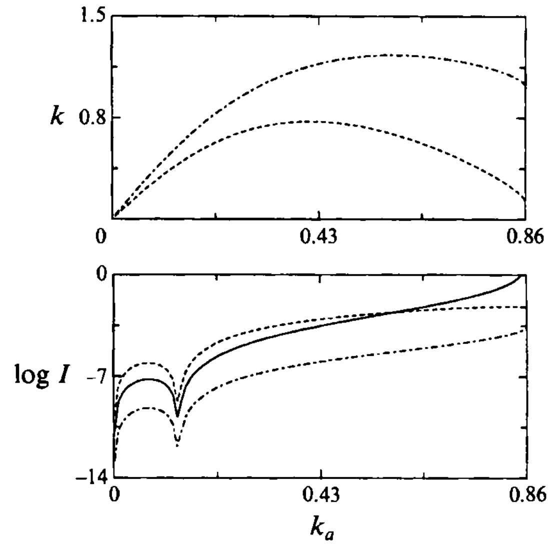

FIGURE 3. Wavenumbers k_b (dashed curve), k_c (dash-dotted curve) and interaction coefficients I_a^{bc} (solid curve), I_b^{ca} (dashed curve), I_c^{ab} (dash-dotted curve) of an explosive resonant triad in the two-layer Phillips model, as a function of the wavenumber k_a. The basic flow parameters are those of figure 2 and the wave meridional indices are $n_a = 0.2l$, $n_b = 0.6l$, $n_c = 0.8l$. Normalization is based on the wave potential enstrophy.

6.3. *Three-layer model*

The linear stability of this model has been studied by Davey (1977), while Romanova (1987) has considered the ERI when the model is unbounded in the meridional direction. Solving the linear eigenvalue problem and using the criteria (6.9) and (6.11), we can investigate the nonlinear stability of a channel model. For $F_1 = F_2 = F_3 \equiv F$ the only parameters describing the system can be denoted

$$K_0^* = K_0/F, \quad U_{s1}^* = F(U_1 - U_2)/\beta, \quad U_{s2}^* = F(U_2 - U_3)/\beta.$$

Straightforward calculation shows that it is impossible to find a layer j satisfying (6.9) or (6.11) if and only if

$$U_{s1}^* > 1 \quad \text{and} \quad 0 < U_{s2}^* < \min(1, U_{s1}^* - 1), \tag{6.17}$$

or

$$-1 < U_{s1}^* < 0 \quad \text{and} \quad U_{s2}^* < U_{s1}^* - 1. \tag{6.18}$$

In these situations, the system is nonlinearly unstable even when it is linearly stable, and ERI plays a leading part in the instability. In other situations, the system is nonlinearly stable as soon as it is linearly stable. It is worth mentioning that Q_{2y} is oppositely signed to Q_{1y} and Q_{3y} when (6.17) or (6.18) is satisfied, while the converse is not true.

To illustrate this conclusion, we show the stability diagrams corresponding to the three-layer model in figure 4. Figures 4(*a*)–4(*d*) display the stability characteristics in the plane (K_0^*, U_{s2}^*) for $U_{s1}^* = 0.5$, 1.5, 3 and -0.5, respectively. The conventions are the same as for the two-layer model; in particular, the unshaded regions are unstable for sufficiently small scales of the disturbances. In figures 4(*b*) and 4(*c*), one sees that such regions extend to infinity as $U_{s2}^* \to 0^-$. This is related to the indefinite increase of the short-wave cut-off, the physical explanation for which is given by Davey (1977). In figure 4(*b, d*), a new type of region appears, represented by hatched zones. It corresponds to the fulfilment of (6.17) (for figures 4*b* and 4*c*) or of (6.18) (for figure

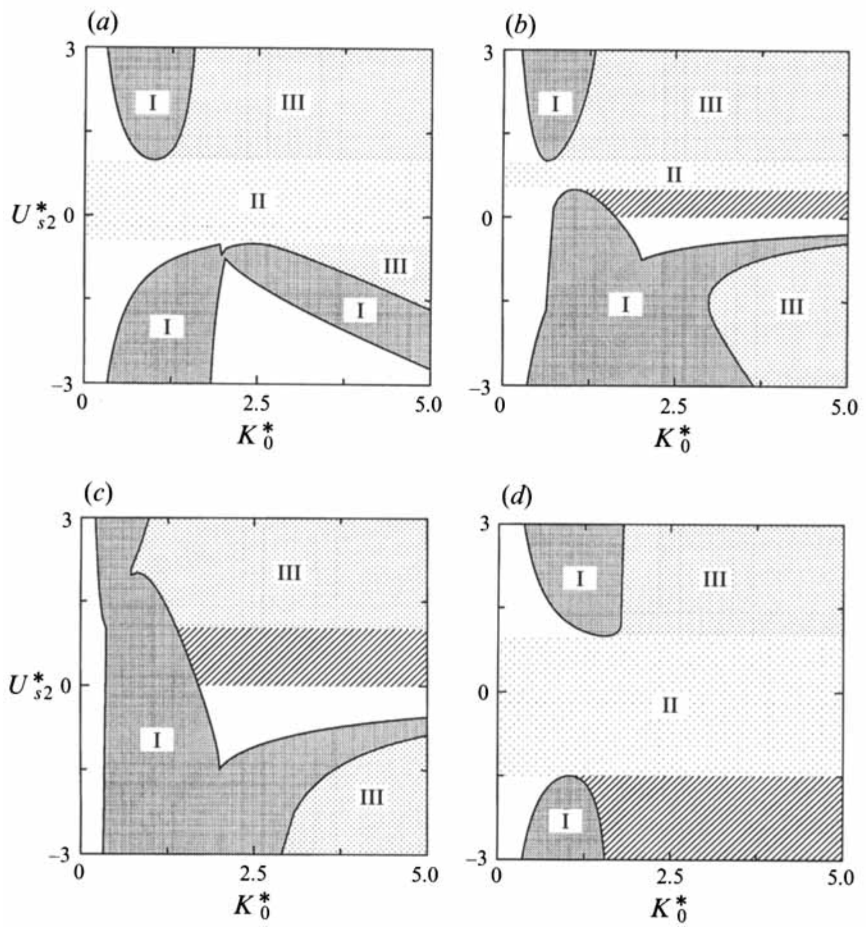

FIGURE 4. (a) Stability diagram of the three-layer Phillips model for $F_1 = F_2 = F_3$ and: (a) $U^*_{s1} = 0.5$, (b) 1.5, (c) 3, (d) −0.5. Region I is linearly unstable for disturbance of wavenumber K_0 and the unshaded region is linearly unstable for smaller-scale disturbances. Regions II and III are proved nonlinearly stable by (6.9) and (6.11), respectively. The hatched regions in (b–d) are unstable through explosive resonant interaction.

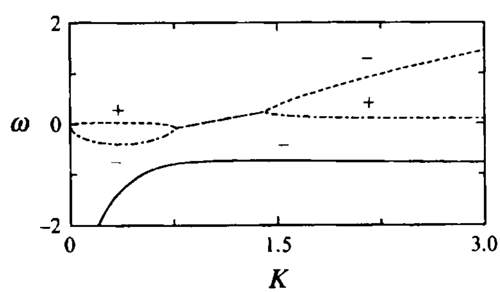

FIGURE 5. Dispersion relation of the three-layer Phillips model for $F_1 = F_2 = F_3 = 1$, $\beta = 1$, $U_1 = 1.5$, $U_2 = 0$ and $U_3 = -0.25$. The sign of the pseudomomentum of each mode is indicated.

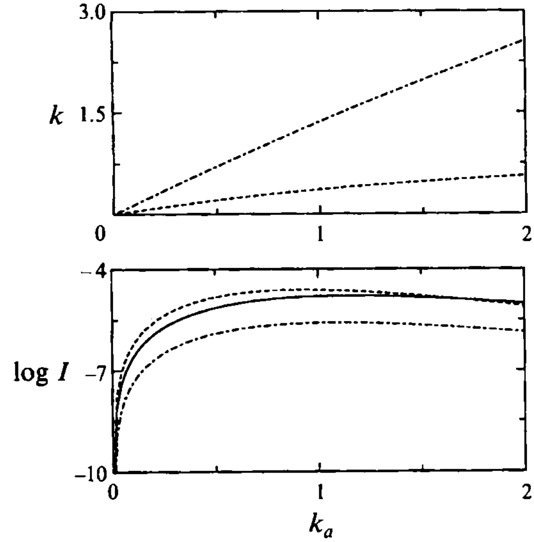

FIGURE 6. Wavenumbers k_b (dashed curve), k_c (dash-dotted curve) and interaction coefficients I_a^{bc} (solid curve), I_b^{ca} (dashed curve), I_c^{ab} (dash-dotted curve) of an explosive resonant resonant triad in the three-layer Phillips model, as a function of the wavenumber k_a. The basic flow parameters are those of figure 5 and the wave meridional indices are $n_a = 1$, $n_b = 3$, $n_c = 4$. Normalization is based on the wave potential enstrophy.

4*d*) in linearly stable conditions. In such region, ERI renders the system nonlinearly unstable. Note that we have confirmed our formulation of the criteria (6.9) and (6.11) by recovering the same stability diagrams by a direct numerical search for an α such that (2.6) or (2.7) is satisfied. The possibility of ERI in linearly stable flow when (6.17) holds is evident in figure 5 which presents the dispersion relation with the sign of the wave pseudomomentum in the case

$$F = 1, \quad \beta = 1, \quad U_1 = 1.5, \quad U_2 = 0, \quad U_3 = -0.25$$

(see also figure 4*b*). Indeed, resonant triads verifying the ERI conditions of §5 are readily found, even if $K_0 > K_{cr} \approx 1.5$. We have numerically searched for such triads *a*, *b* and *c* in the case of a linearly stable channel with $l = 0.4$. The meridional indices have been taken to be

$$n_a = 1, \quad n_b = 3, \quad n_c = 4$$

and the wavenumbers k_b and k_c, as well as the interaction coefficients, are displayed in figure 6 as functions of k_a. Each wave corresponds to a distinct branch in the dispersion diagram shown in figure 5 and the correspondence is easily seen as we have used the same line styles in both figures. For instance, *c* is the wave with the largest wavenumber magnitude and has a positive pseudomomentum. The interaction coefficients clearly possess a maximum for $k_a, k_c \approx 1$ while $k_b \approx 0.3$. The fastest growing disturbances can thus be expected to have lengthscales of this order. However, it is interesting to note that ERI may occur for waves with arbitrarily large wavenumbers. The short-wave cut-off induced by the β-effect is thus only a linear feature when (6.17) or (6.18) is satisfied. Figure 7 displays the analogue of figure 6 for

$$U_1 = -0.5, \quad U_2 = 0, \quad U_3 = 2,$$

the other parameters being unchanged. ERI can then occur in linearly stable condition

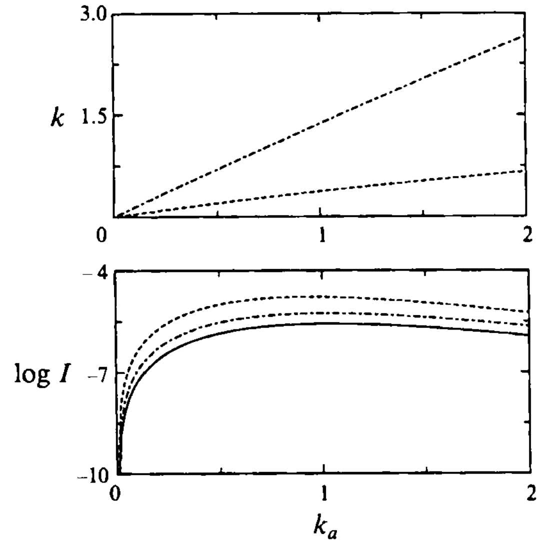

FIGURE 7. Same as figure 6 but with the basic flow parameters $U_1 = -0.5$, $U_2 = 0$ and $U_3 = 2$.

as criterion (6.18) is fulfilled (see also figure 4*d*). Although the shear condition is fairly different, the interaction coefficients are very similar to those found in the preceding example.

7. Discussion

This paper addresses the question of nonlinear interaction between baroclinic Rossby waves in multilayer quasi-geostrophic flow. Greatest attention is paid to waves propagating on a parallel shear flow, and to the role played by the interaction on the stability of the basic flow. The eigenvalue problem satisfied by the normal modes is discussed in detail, and a bound on the phase velocity of unstable modes is derived from the vanishing of pseudomomentum (equation (3.4)). The equations describing nonlinear interactions are deduced using a rigorous expansion of the basic equations in normal modes, with the orthogonality relations in the sense of pseudomomentum (Held 1985) as projection operators. The expansion is sufficiently general to involve stable, unstable, and marginally stable modes, and is thus well suited to analyse the interactions in a linearly unstable flow. Properties of the interaction coefficients are deduced from the nonlinear conservation laws of pseudoenergy and pseudomomentum without manipulation of the explicit expression for the coefficients. This demonstrates the arbitrariness of the usual classification between conservative and non-conservative interactions (e.g. Craik 1985): in a dissipationless fluid, the interactions may always be seen as conservative, provided that the *ad hoc* quantity is considered.

In the case of resonant triads of baroclinic Rossby waves, the properties of these interaction coefficients permit the discussion of two important phenomena: wave instability and explosive resonant interaction (ERI). The latter phenomenon occurs in resonant triads where the wave with the largest wavenumber has oppositely signed pseudomomentum to the other two. It constitutes a nonlinear instability mechanism capable of amplifying infinitesimal disturbances. We discuss its importance in connection with the recent results on the nonlinear stability of multilayer quasi-geostrophic flow due to Mu Mu (1991), Ripa (1992, 1993) and Mu Mu *et al.* (1994).

We then consider the generalized Phillips model, i.e. a multilayer model with constant velocity in each layer, and derive a particularized formulation of the sufficient stability conditions. The system is proved to be nonlinearly stable if it is linearly stable and if there exists a layer j such that $Q_{iy}^{-1}(U_i - U_j)$ is sign definite for all layers $i \neq j$. Moreover, we show that ERI necessarily occurs in the flow if this condition is not satisfied. The stability criterion is therefore a necessary and sufficient condition of nonlinear stability.

For the two-layer Phillips model, the result of Mu Mu *et al.* (1994), namely the equivalence of linear and nonlinear stability, is recovered. Consequently, ERI may only occur when the flow is in linearly supercritical conditions. However, it can play a significant role in that case by amplifying waves which are predicted stable by linear theory. In the three-layer model, situations are found where the flow is linearly stable and ERI of baroclinic Rossby waves acts as nonlinear destabilizing mechanism. ERI is hence in this case of crucial importance in baroclinic instability. In this respect, it would be interesting to examine the saturation of the instability generated through ERI. Of course, the triad interaction hypothesis breaks down long before the explosion time and a larger number of amplitude equation would be necessary to follow the long-term evolution of the instability. An alternative approach to a numerical solution would be the use of the powerful technique developed by Shepherd (1988, 1993) to study the saturation of the linear instability in the two-layer model.

Another point worthy of attention is the ERI involving one or more waves near marginal stability. We have mentioned that such a wave cannot be isolated from the wave with which it coalesces at the marginally stable point, and that the new variables recently proposed by Romanova (1994) must be introduced. The behaviour of such a system involving at least four amplitude equations should be investigated to decide whether it involves instability of the basic flow. The question of interacting triads partially constituted of unstable modes also seems of great interest, as it may have direct consequences on the nonlinear evolution of baroclinic instability.

The research reported on in this paper could be extended to other models of geophysical interest, in particular those for which nonlinear stability conditions have been established. The two-layer non-geostrophic model, whose linear stability has been carefully studied by Sakaï (1989), could be considered using Ripa's general stability conditions (Ripa 1991). These conditions only ensure formal stability but, as already mentioned this is sufficient to preclude ERI. It would also be interesting to examine the interactions between Rossby waves in Eady's model of baroclinic instability. Mu Mu & Shepherd (1994) have recently found a nonlinear stability condition which differs significantly from the linear one. One may expect ERI to play a role in Eady's model similar to one explained in the present paper. However, there seems to be no doubt that critical levels will prove fundamental in such a continuous model; the nonlinear interaction involving packets of singular modes is thus a preliminary point to consider.

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Appendix. Demonstration of the matrix property (6.7)

Consider a symmetric $N \times N$ matrix $\boldsymbol{M}$ and the vector $\boldsymbol{q}$ such that

$$\boldsymbol{M}\boldsymbol{q} = 0. \tag{A 1}$$

The elements of $\boldsymbol{M}$ may be complex, and it is important to note that $\boldsymbol{M} = \boldsymbol{M}^T$ is required and not $\boldsymbol{M} = \boldsymbol{M}^*$ (in our case of interest, only the diagonal elements of $\boldsymbol{M}$ are complex). We shall show that

$$\frac{[M]_k}{q_k^2} = \frac{[M]_1}{q_1^2}, \quad k = 2, N, \tag{A 2}$$

where $[M]_k$ is the minor of the diagonal element k of $\boldsymbol{M}$. Denoting by m_{ij} and q_j generic elements of $\boldsymbol{M}$ and $\boldsymbol{q}$, respectively, we decompose (A 1) into the three relations

$$m_{11}\, q_1 + m_{1k}\, q_k + \boldsymbol{m}_1^T \bar{\boldsymbol{q}} = 0, \tag{A 3a}$$

$$m_{k1}\, q_1 + m_{kk}\, q_k + \boldsymbol{m}_k^T \bar{\boldsymbol{q}} = 0, \tag{A 3b}$$

$$\boldsymbol{m}_1 q_1 + \boldsymbol{m}_k q_k + \bar{\boldsymbol{M}} \bar{\boldsymbol{q}} = 0, \tag{A 3c}$$

where we have introduced the vectors $\boldsymbol{m}_1 = (m_{j1})^T$, $\boldsymbol{m}_k = (m_{jk})^T$, $\bar{\boldsymbol{q}} = (q_j)^T, j \neq 1, k$ and the $(N-2) \times (N-2)$ matrix $\bar{\boldsymbol{M}}$ which is obtained from $\boldsymbol{M}$ by removing the rows and columns 1 and k. The symmetry of $\boldsymbol{M}$ has been used to write (A 3). From (A 3c) we can express $\bar{\boldsymbol{q}}$ as a function of q_1 and q_k. Introducing this expression in (A 3a) and (A 3b) and multiplying the resulting equations, we find

$$q_1^2(m_{11} - \boldsymbol{m}_1^T \bar{\boldsymbol{M}}^{-1} \boldsymbol{m}_1) = q_k^2(m_{kk} - \boldsymbol{m}_k^T \bar{\boldsymbol{M}}^{-1} \boldsymbol{m}_k). \tag{A 4}$$

It is now easy to prove that

$$(m_{kk} - \boldsymbol{m}_k^T \bar{\boldsymbol{M}}^{-1} \boldsymbol{m}_k) = (\det \bar{\boldsymbol{M}})^{-1} [M]_1 \tag{A 5}$$

by noting the successive equalities

$$[M]_1 = \det \begin{pmatrix} m_{kk} & \boldsymbol{m}_k^T \\ \boldsymbol{m}_k & \bar{\boldsymbol{M}} \end{pmatrix} = \det \left\{ \begin{pmatrix} 1 & \boldsymbol{m}_k^T \\ 0 & \bar{\boldsymbol{M}} \end{pmatrix} \begin{pmatrix} m_{kk} - \boldsymbol{m}_k^T \bar{\boldsymbol{M}}^{-1} \boldsymbol{m}_k & 0 \\ \bar{\boldsymbol{M}}^{-1} \boldsymbol{m}_k & 1 \end{pmatrix} \right\}.$$

Introducing (A 5) and the analogous relation for $(m_{11} - \boldsymbol{m}_1^T \bar{\boldsymbol{M}}^{-1} \boldsymbol{m}_1)$ in (A 4) leads then to (A 2).

REFERENCES

BECKER, J. M. & GRIMSHAW, R. H. J. 1993 Explosive resonant triads in a continuously stratified shear flow. *J. Fluid Mech.* **257**, 219–228.

CAIRNS, R. A. 1979 The role of negative energy waves in some instabilities of parallel flow. *J. Fluid Mech.* **92**, 1–14.

CRAIK, A. D. D. 1985 *Wave Interactions and Fluid Flows*. Cambridge University Press.

CRAIK, A. D. D. & ADAM, J. A. 1979 Explosive resonant wave interaction in a three-layer fluid flow. *J. Fluid Mech.* **92**, 15–33.

DAVEY, M. K. 1977 Baroclinic instability in a fluid with three layers. *J. Atmos. Sci.* **34**, 1224–1234.

HASSELMANN, K. 1967 A criterion for nonlinear wave stability. *J. Fluid Mech.* **30**, 737–739.

HELD, I. M. 1985 Pseudomomentum and the orthogonality of modes in shear flows. *J. Atmos. Sci.* **42**, 2280–2288.

HOLM, D. D., MARSDEN, J. E., RATIU, T. & WEINSTEIN, A. 1985 Nonlinear stability of fluid and plasma equilibria. *Phys. Rep.* **123**, 1–116.

JONES, S. 1979*a* Rossby waves interactions and instabilities in a rotating two-layer fluid on a beta-plane. Part I: resonant interactions. *Geophys. Astrophys. Fluid Dyn.* **11**, 289–322.

JONES, S. 1979*b* Rossby waves interactions and instabilities in a rotating two-layer fluid on a beta-plane. Part II: stability. *Geophys. Astrophys. Fluid Dyn.* **12**, 1–33.

LIU YONGMING & MU MU 1992 A problem related to nonlinear stability criteria for multilayer quasi-geostrophic flow. *Adv. Atmos. Sci.* **9**, 337–345.

McIntyre, M. E. & Shepherd, T. G. 1987 An exact local conservation theorem for finite-amplitude disturbances to non-parallel shear flows, with remarks on Hamiltonian structure and on Arnol'd's stability theorems. *J. Fluid Mech.* **181**, 527–565.

Mu Mu 1991 Nonlinear stability criteria for motions of multilayer quasi-geostrophic flow. *Science in China* **B34**, 1516–1528.

Mu Mu, Zeng Qingcun, Shepherd, T. G. & Liu Yongming 1994 Nonlinear stability of multilayer quasi-geostrophic flow. *J. Fluid Mech.* **264**, 165–184.

Mu Mu & Shepherd, T. G. 1994 Nonlinear stability of Eady's model. *J. Atmos. Sci.* **51**, 3427–3436.

Pedlosky, J. 1987 *Geophysical Fluid Dynamics*. Springer.

Phillips, N. A. 1954 Energy transformation and meridional circulation associated with simple baroclinic waves in two-level quasi-geostrophic model. *Tellus* **3**, 273–286.

Ripa, P. 1983 Weak interactions of equatorial waves in a one-layer model. Part I: general properties. *J. Phys. Oceanogr.* **13**, 1208–1226.

Ripa, P. 1990 Positive, negative and zero wave energy and the flow stability problem in the Eulerian and Lagrangian–Eulerian descriptions. *Pure Appl. Geophys.* **133**, 713–732.

Ripa, P. 1991 General stability conditions for a multilayer model. *J. Fluid Mech.* **222**, 117–139.

Ripa, P. 1992 Wave energy-momentum and pseudoenergy-momentum conservation for the layered quasi-geostrophic instability problem. *J. Fluid Mech.* **235**, 379–398.

Ripa, P. 1993 Arnol'd's second stability theorem for the equivalent barotropic model. *J. Fluid Mech.* **257**, 597–605.

Romanova, N. N. 1987 Explosive instability in a three-layered rotating liquid. *Izv. Atmos. Ocean. Phys.* **23**, 269–274.

Romanova, N. N. 1992 Construction of normal variables for waves in an unstable *n*-layer moving medium. *Izv. Atmos. Ocean. Phys.* **28**, 343–350.

Romanova, N. N. 1994 Hamiltonian description of weakly nonlinear wave dynamics in non-equilibrium media. *Dokl. Akad. Nauk.* **335**, 592–594 (in Russian).

Sakaï, S. 1989 Rossby–Kelvin instability: a new type of ageostrophic instability caused by a resonance between Rossby waves and gravity waves. *J. Fluid Mech.* **202**, 149–176.

Shepherd, T. G. 1988 Nonlinear saturation of baroclinic instability. Part I: the two-layer model. *J. Atmos. Sci.* **45**, 2014–2025.

Shepherd, T. G. 1993 Nonlinear saturation of baroclinic instability. Part III: bounds on the energy. *J. Atmos. Sci.* **50**, 2697–2709.

Tsutahara, M. 1984 Resonant interaction of internal waves in a stratified shear flow. *Phys. Fluids* **27**, 1942–1947.

Tung, K. K. 1983 Initial-value problem for Rossby waves in a shear flow with critical level. *J. Fluid Mech.* **133**, 443–469.

Vanneste, J. & Vial, F. 1995 On the nonlinear interaction of geophysical waves in shear flows. *Geophys. Astrophys. Fluid Dyn.* in press (referred to herein as VV).